

The derived algebra of a stabilizer, families of coadjoint orbits and sheets

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Abstract

Let \mathfrak{g} be a finite-dimensional real or complex Lie algebra and $\mu \in \mathfrak{g}^*$. In the first part of the paper we discuss the relation between the derived algebra of the stabilizer of μ and the set of coadjoint orbits, which have the same dimension as the orbit of μ . In the second part of the paper we consider a real or complex semisimple Lie algebra \mathfrak{g} and discuss the relation between the derived algebra of the centralizer of an element $\mu \in \mathfrak{g}$ and sheets of \mathfrak{g} . In particular, we prove that if \mathfrak{g} is either $\mathfrak{sl}(n, \mathbb{C})$ or a compact real Lie algebra, then the derived algebra of the centralizer of $\mu \in \mathfrak{g}$ is the orthogonal complement to the sheet through μ .

1 Introduction

Let \mathfrak{g} be a finite-dimensional real or complex Lie algebra. The group G acts on the dual space \mathfrak{g}^* via the coadjoint action, and \mathfrak{g}^* is foliated into the orbits of this action. Let us consider the union of all orbits, which have the same codimension k . Denote this union by \mathfrak{g}_k^* .

The study of the sets \mathfrak{g}_k^* was initiated by A.Kirillov (see [5]) in relation with the “method of orbits”, which relates the unitary dual of G to the set of coadjoint orbits \mathfrak{g}^*/G . The sets \mathfrak{g}_k^*/G appear in this picture as the natural strata of \mathfrak{g}^*/G , therefore it is important to understand the geometry of \mathfrak{g}_k^* for each k .

On the other hand, we are motivated by the fact that the sets \mathfrak{g}_k^* arise when one considers certain bihamiltonian systems on \mathfrak{g}^* (such as the systems obtained by the argument shift method).

The aim of the paper is to prove the following simple geometric observation: *any vector which is tangent to \mathfrak{g}_k^* at point μ vanishes on the derived algebra of the stabilizer of μ .* Our second goal is to understand whether all vectors vanishing on the derived algebra of the stabilizer of μ are tangent to \mathfrak{g}_k^* at point μ , i.e. *is the tangent space to \mathfrak{g}_k^* at point μ equal to the annihilator of the derived algebra of the stabilizer of μ ?* In general, this is not the case. However, this situation seems to be quite common.

In the semisimple case the sets \mathfrak{g}_k^* can be identified, via the Killing form, with the sets of adjoint orbits of codimension k . The irreducible components

of these latter sets are called sheets. The geometry of sheets is nowadays quite well understood (see [8] and references therein).

Our statement in this case can be reformulated as follows: *the derived algebra of the centralizer of μ is orthogonal to any sheet passing through μ* . We also prove that *for any $\mu \in \mathfrak{sl}(n, \mathbb{C})$ the derived algebra of the centralizer of μ is exactly the orthogonal complement to the sheet passing through μ* . This is also the case for all compact Lie algebras. We conjecture that this statement remains true for all classical Lie algebras (provided that there is a unique sheet passing through a given point, which is not always the case).

2 The general case: the derived algebra of a stabilizer versus families of coadjoint orbits

2.1 Families of coadjoint orbits of the same dimension

Let \mathfrak{g} be a real or complex Lie algebra. Let $\mathfrak{g}_\mu = \{a \in \mathfrak{g} \mid \text{ad}_a^* \mu = 0\}$ be the stabilizer of an element $\mu \in \mathfrak{g}^*$ for the coadjoint action.

Denote by \mathfrak{g}_k^* the set of elements $\mu \in \mathfrak{g}^*$ such that $\dim \mathfrak{g}_\mu = k$. It is clear that \mathfrak{g}_k^* is an algebraic variety for each k , and \mathfrak{g}^* is the disjoint union of all \mathfrak{g}_k^* .

\mathfrak{g}_k^* can be also defined as the union of all coadjoint orbits of codimension k .

Let us denote \mathfrak{g}_k^* , where $k = \dim \mathfrak{g}_\mu$, by $\mathfrak{g}^*(\mu)$. In other words, $\mathfrak{g}^*(\mu)$ is \mathfrak{g}_k^* passing through μ .

2.2 The main statement

Proposition 2.1. *Let γ be a smooth curve such that $\gamma(0) = \mu$ and $\gamma(t) \in \mathfrak{g}^*(\mu)$ for all t . Then the tangent vector $\dot{\gamma}(0)$ vanishes on the derived algebra of \mathfrak{g}_μ :*

$$\langle \dot{\gamma}(0), [\mathfrak{g}_\mu, \mathfrak{g}_\mu] \rangle = 0.$$

Proof. Since $\dim \mathfrak{g}_{\gamma(t)} = \dim \mathfrak{g}_\mu$ for all t , it is possible to choose smooth $e_1(t), \dots, e_k(t)$, which generate $\mathfrak{g}_{\gamma(t)}$. Since $e_i(t)$ are elements of the stabilizer, we have

$$\langle \gamma(t), [e_i(t), e_j(t)] \rangle = 0.$$

Differentiating with respect to t , we get

$$\langle \dot{\gamma}(0), [e_i(0), e_j(0)] \rangle + \langle \mu, [\dot{e}_i(0), e_j(0)] \rangle + \langle \mu, [e_i(0), \dot{e}_j(0)] \rangle = 0,$$

but since $e_i(t)$ are elements of the stabilizer, the last two terms are equal to zero, and we have

$$\langle \dot{\gamma}(0), [e_i(0), e_j(0)] \rangle = 0.$$

Since $e_i(t)$ generate $\mathfrak{g}_{\gamma(t)}$, the tangent vector $\dot{\gamma}(0)$ vanishes on the whole derived algebra of \mathfrak{g}_μ . \square

Remark 2.1. The proposition remains true if $\gamma(t)$ is only defined for $t \geq 0$ and the right derivative $\dot{\gamma}_+(0)$ exists (this may happen if $\mathfrak{g}^*(\mu)$ has a kind of cuspidal singularity at μ).

Corollary 2.1. *Consider the case when $\mathfrak{g}^*(\mu)$ is smooth at μ . Then*

1. *Each element of the tangent space to $\mathfrak{g}^*(\mu)$ at the point μ vanishes on the derived algebra of \mathfrak{g}_μ :*

$$\langle T_\mu \mathfrak{g}^*(\mu), [\mathfrak{g}_\mu, \mathfrak{g}_\mu] \rangle = 0.$$

2. *The following inequality is satisfied:*

$$\dim[\mathfrak{g}_\mu, \mathfrak{g}_\mu] \leq \text{codim}_\mu \mathfrak{g}^*(\mu). \quad (1)$$

3. *The equality*

$$\dim[\mathfrak{g}_\mu, \mathfrak{g}_\mu] = \text{codim}_\mu \mathfrak{g}^*(\mu)$$

is satisfied if and only if $T_\mu \mathfrak{g}^(\mu)$ is exactly the annihilator of $[\mathfrak{g}_\mu, \mathfrak{g}_\mu]$, i.e. the set of $\nu \in \mathfrak{g}^*$ such that $\langle \nu, [\mathfrak{g}_\mu, \mathfrak{g}_\mu] \rangle = 0$.*

Remark 2.2. The inequality (1) is in some sense similar to the inequality for the index of a stabilizer: the index of the stabilizer \mathfrak{g}_μ of an element under coadjoint action is greater or equal than the index of \mathfrak{g} (Vinberg, see [9]).

Corollary 2.2. *The codimension of the set of elements $\mu \in \mathfrak{g}^*$ such that*

$$\dim[\mathfrak{g}_\mu, \mathfrak{g}_\mu] \geq k$$

is at least k .

Example 2.1. For regular μ we obtain a well-known fact: \mathfrak{g}_μ is abelian.

Example 2.2 (Extended argument shift method). Assume that we want to find a complete family of polynomials in involution on \mathfrak{g}^* . In many cases this can be done by the argument shift method (Mischenko-Fomenko, [3]). However, this method doesn't work if the set of singular elements is a hypersurface (Bolsinov, [1]). Using our inequality, we see that in this case we have $\dim[\mathfrak{g}_\mu, \mathfrak{g}_\mu] \leq 1$ for almost all singular elements, which means that the stabilizer \mathfrak{g}_μ of a general position singular element has one of the following types:

1. Abelian.
2. $\mathfrak{aff}(1) \oplus \text{abelian}$, where $\mathfrak{aff}(1)$ is the Lie algebra of affine transformations of the line.
3. $\mathfrak{h}_n \oplus \text{abelian}$, where \mathfrak{h}_n is the n -dimensional Heisenberg algebra (and n is odd).

Since the set of singular elements is a hypersurface, the union of its irreducible components of maximal dimension is given by a polynomial equation $D(x) = 0$.

One can show that if the stabilizer of a general position singular element is isomorphic to $\mathfrak{aff}(1) \oplus \mathbb{K}^{\text{ind } \mathfrak{g}}$, then the shifts of D together with the shifts of invariants form a complete family of polynomials in involution.

The inequality (1) can be rewritten as

$$\dim_{\mu} \mathfrak{g}^*(\mu) - \dim O(\mu) \leq \dim \mathfrak{g}_{\mu} - \dim[\mathfrak{g}_{\mu}, \mathfrak{g}_{\mu}], \quad (2)$$

where $O(\mu)$ is the coadjoint orbit of μ .

Coadjoint orbits form families and the difference $\dim_{\mu} \mathfrak{g}^*(\mu) - \dim O(\mu)$ is exactly the (local) dimension of such a family. The inequality (2) estimates this dimension.

Example 2.3. Let $\mathfrak{g} = \mathfrak{gl}(n)$,

$$\mu = \text{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{k_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2}, \dots, \underbrace{\lambda_s, \dots, \lambda_s}_{k_s}).$$

We have $\mathfrak{g}_{\mu} \simeq \mathfrak{gl}(k_1) \oplus \mathfrak{gl}(k_2) \oplus \dots \oplus \mathfrak{gl}(k_s)$, so $\dim \mathfrak{g}_{\mu} - \dim[\mathfrak{g}_{\mu}, \mathfrak{g}_{\mu}] = s$. On the other hand, the orbits close to $O(\mu)$ which have the same dimension are parameterized by the eigenvalues $\lambda_1, \dots, \lambda_s$, which means that $\dim_{\mu} \mathfrak{g}^*(\mu) - \dim O(\mu)$ is also equal to s and the inequality (2) turns into equality. We will show this happens every time the transverse Poisson structure to a coadjoint orbit is linearizable.

2.3 What happens if the transverse Poisson structure is linearizable

Recall that, by Weinstein's theorem (see [12]), for any Poisson manifold M and any point $\mu \in M$ it is possible, in the neighborhood of μ , to decompose M into a direct product of a symplectic manifold and a Poisson manifold with a Poisson structure vanishing at μ . This latter Poisson structure is unique up to a diffeomorphism and is called the transverse Poisson structure (to the symplectic leaf) at the point μ .

In the case when our Poisson manifold is the dual \mathfrak{g}^* to a Lie algebra \mathfrak{g} , then the symplectic leaves are exactly the coadjoint orbits, so we can speak about the transverse Poisson structure to the coadjoint orbit. The linear part of the transverse Poisson structure at a point μ is the Lie-Poisson structure of the stabilizer \mathfrak{g}_{μ} . Consequently, if the transverse Poisson structure at μ is linearizable, then the Lie-Poisson structure in the neighborhood of μ can be decomposed into a direct product of a symplectic structure and the Lie-Poisson structure of the stabilizer \mathfrak{g}_{μ} .

Proposition 2.2. *Assume that the transverse Poisson structure to a coadjoint orbit at a point μ is linearizable. Then $\mathfrak{g}^*(\mu)$ is smooth at μ and*

$$\dim_{\mu} \mathfrak{g}^*(\mu) - \dim O(\mu) = \dim \mathfrak{g}_{\mu} - \dim[\mathfrak{g}_{\mu}, \mathfrak{g}_{\mu}].$$

Proof. Since the transverse Poisson structure at μ is linearizable, \mathfrak{g}^* can be locally decomposed, together with the Poisson bracket, into a direct product of \mathfrak{g}_μ^* and a symplectic manifold W . Consequently, each element ν sufficiently close to μ may be written as a pair (ξ, ψ) , where $\xi \in \mathfrak{g}_\mu^*, \psi \in W$. The dimension of \mathfrak{g}_ν is equal to the dimension of the stabilizer of ξ in \mathfrak{g}_μ^* , hence $\mathfrak{g}^*(\mu)$ consists of such pairs (ξ, ψ) , that the stabilizer of ξ in \mathfrak{g}_μ^* coincides with \mathfrak{g}_μ^* . This means that ξ vanishes on the derived algebra and $\dim \mathfrak{g}^*(\mu) = \dim W + \dim \mathfrak{g}_\mu - \dim[\mathfrak{g}_\mu, \mathfrak{g}_\mu]$. But $\dim W$ is exactly the dimension of the orbit, which proves our proposition. \square

Example 2.4 (M. Duflo, see [11], [2]). Let \mathfrak{g} be a Lie algebra, given by the following linear Poisson structure:

$$\frac{\partial}{\partial x_1} \wedge (x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + 2x_4 \frac{\partial}{\partial x_4}) + x_4 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$

If we take $x_4 = 0$ and $x_2^2 + x_3^2 > 0$, then the stabilizer is abelian, which means that $\dim[\mathfrak{g}_\mu, \mathfrak{g}_\mu] = 0$. On the other hand, we have $\text{codim } \mathfrak{g}^*(\mu) = 1$. Consequently, the transverse Poisson structure is not linearizable.

3 The semisimple case: the derived algebra of a centralizer versus sheets

3.1 Sheets

In the semisimple case the coadjoint and adjoint actions can be identified via the Killing form. This identification maps the sets \mathfrak{g}_k^* to the sets $\mathfrak{g}^{(k)}$, where $\mathfrak{g}^{(k)}$ is the set of all $\mu \in \mathfrak{g}$ such that the dimension of the centralizer \mathfrak{g}^μ of μ is exactly k .

The irreducible components of the sets $\mathfrak{g}^{(k)}$ are called sheets of \mathfrak{g} .

Let us recall some facts about the topology of sheets:

1. The sheets are not smooth in general. However, if \mathfrak{g} is a *classical* semisimple Lie algebra, then sheets are smooth (Im Hof, [4]).
2. The sheets are not necessarily disjoint even in the classical case. However, they are disjoint for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ (Kraft and Luna, [6], Peterson, [10]).

We are going to study the relation between sheets and the derived algebra of a centralizer.

3.2 The main statement in the semisimple case

Using Corollary 2.1, we obtain the following

Proposition 3.1. *Let \mathfrak{g} be a real or complex semisimple Lie algebra. Suppose that $\mu \in \mathfrak{g}$ belongs to a sheet S , and S is smooth at the point μ . Then*

- The derived algebra of the centralizer of μ is orthogonal to S at the point μ :

$$\langle [\mathfrak{g}^\mu, \mathfrak{g}^\mu], T_\mu S \rangle = 0.$$

- $\dim [\mathfrak{g}^\mu, \mathfrak{g}^\mu] \leq \text{codim } S$.
- $\dim \mathfrak{g}^\mu - \dim [\mathfrak{g}^\mu, \mathfrak{g}^\mu] \geq \dim S - \dim O(\mu)$ where $O(\mu)$ is the adjoint orbit of μ .
- The following three statements are equivalent:

1. The derived algebra of the centralizer of μ is exactly the orthogonal complement to S at μ :

$$[\mathfrak{g}^\mu, \mathfrak{g}^\mu] = (T_\mu S)^\perp. \quad (3)$$

2. The dimension of the derived algebra of the centralizer of μ is equal to the codimension of S :

$$\dim [\mathfrak{g}^\mu, \mathfrak{g}^\mu] = \text{codim } S.$$

3. The codimension of the derived algebra in the centralizer of μ is equal to the codimension of the orbit of μ in S :

$$\dim \mathfrak{g}^\mu - \dim [\mathfrak{g}^\mu, \mathfrak{g}^\mu] = \dim S - \dim O(\mu).$$

If the equality (3) holds, we have a very nice picture: the centralizer is the orthogonal complement to the orbit while the derived algebra of it is the orthogonal complement to the sheet (see Fig. 1).

3.3 The case of a semisimple element

Proposition 3.2. *If \mathfrak{g} is a (real or complex) semisimple Lie algebra and $\mu \in \mathfrak{g}$ is semisimple, then there is only one sheet S passing through μ , S is smooth at μ and the equality (3) holds.*

Proof. Taking into account Proposition 2.2 it suffices to show that the transverse Poisson structure at a semisimple point is linearizable. This can be done using the following theorem ([7]):

If the centralizer of μ has a complement \mathfrak{m} such that $[\mathfrak{g}^\mu, \mathfrak{m}] \subset \mathfrak{m}$, then the transverse Poisson structure is linearizable.

Since μ is semisimple, the centralizer of it is a Cartan subalgebra \mathfrak{h} plus all the root spaces corresponding to the roots vanishing on μ . If we define \mathfrak{m} as the sum of all other root spaces, the condition of the theorem will be satisfied, which proves our proposition. \square

Corollary 3.1. *Let \mathfrak{g} be a compact real Lie algebra. Then all sheets of \mathfrak{g} are smooth, disjoint and the equality (3) holds for every $\mu \in \mathfrak{g}$.*

Proof. Indeed, all elements of a compact algebra are semisimple. \square

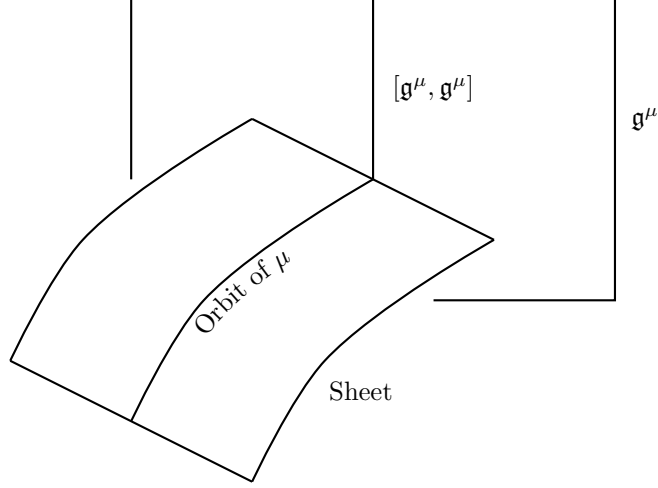


Figure 1

3.4 The case of $\mathfrak{sl}(n, \mathbb{C})$

Proposition 3.3. *The equality (3) holds for every $\mu \in \mathfrak{sl}(n, \mathbb{C})$.*

Proof. Denote by $h(\lambda)$ the size of the largest Jordan block of μ with the eigenvalue λ . We are going to prove that

$$\dim \mathfrak{g}^\mu - \dim [\mathfrak{g}^\mu, \mathfrak{g}^\mu] = \dim S - \dim O(\mu) = \left(\sum h(\lambda) \right) - 1.$$

1. $\dim \mathfrak{g}^\mu - \dim [\mathfrak{g}^\mu, \mathfrak{g}^\mu] = \left(\sum h(\lambda) \right) - 1$.

It suffices to show that for every nilpotent $\nu \in \mathfrak{gl}(m, \mathbb{C})$ the difference $\dim \mathfrak{g}^\nu - \dim [\mathfrak{g}^\nu, \mathfrak{g}^\nu]$ is equal to the size of the largest Jordan block. This can be easily done by studying the commutation relations for \mathfrak{g}^ν given by Oksana Yakimova in [13] (see Section “Basis of a centralizer”).

2. $\dim S - \dim O(\mu) = \left(\sum h(\lambda) \right) - 1$.

This fact is known for a nilpotent element (Anne Moreau, see Lemma 3.2 in [8]). Let us give a sketch of the proof for an arbitrary element.

For each eigenvalue λ take a sequence of complex numbers $\varepsilon_1(\lambda), \dots, \varepsilon_{h(\lambda)}(\lambda)$ and consider an element obtained from μ by adding a diagonal matrix to each Jordan block with numbers $\varepsilon_1(\lambda), \dots, \varepsilon_k(\lambda)$ on the diagonal, where λ is the eigenvalue corresponding to the block and k is the size of block. Now we have a family $\mu_\varepsilon \in \mathfrak{sl}(n, \mathbb{C})$ of the dimension $\left(\sum h(\lambda) \right) - 1$. It is quite easy to check that the dimension of the centralizer of μ_ε is equal to the dimension of the centralizer of μ . Since the family μ_ε is smooth, all

μ_ε belong to the sheet S passing through μ . Since only finite number of μ_ε are conjugated to each other, we have

$$\dim S \geq \dim O(\mu) + \left(\sum h(\lambda) \right) - 1.$$

On the other hand, by proposition 3.1,

$$\dim S - \dim O(\mu) \leq \dim \mathfrak{g}^\mu - \dim [\mathfrak{g}^\mu, \mathfrak{g}^\mu] = \left(\sum h(\lambda) \right) - 1,$$

which proves our proposition. □

3.5 The case of an arbitrary classical semisimple Lie algebra

Conjecture 1. *Let \mathfrak{g} be a real or complex classical semisimple Lie algebra.*

1. *If there is only one sheet S passing through $\mu \in \mathfrak{g}$, then the equality (3) holds, i.e.*

$$[\mathfrak{g}^\mu, \mathfrak{g}^\mu] = (\mathrm{T}_\mu S)^\perp.$$

2. *If $\mu \in \mathfrak{g}$ belongs to several sheets S_1, \dots, S_k , then*

$$[\mathfrak{g}^\mu, \mathfrak{g}^\mu] = \left(\sum \mathrm{T}_\mu S_i \right)^\perp.$$

This conjecture was proved for rigid nilpotent orbits (i.e. orbits which are at the same time sheets) by Yakimova in [13].

Also note that the conjecture is false in general for exceptional Lie algebras (see example of Yakimova for G_2 in [13], Remark 3).

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